授業資料（12 月 4 日，7 日）

『AD LOCOS PLANOS ET SOLIDOS』

（CEUVRES DE FERMAT の表紙）

ŒUVRES

DE FERMAT

PUBLIÉES PAR LES SOINS DE

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SOUS LES AUSPICES

DU MINISTÈRE DE L'INSTRUCTION PUBLIQUE.

TOME PREMIER.

ŒUVRES MATHÉMATIQUES DIVERSES. — OBSERVATIONS SUR DIOPHANTE.

PARIS,

GAUTHIER-VILLARS ET FILS, IMPRIMEURS-LIBRAIRES

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE

Quai des Grands-Augustins, 55.

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ピエル・ド・フェルマー

フェルマーの外面的生活は静かで規律正しく波乱のないものだったが，数学的業績は独創性や深遠さ，多様性において傑出していた．実際，彼は全時代を通じて最も偉大な「純粋」数学者の一人である．

1601 年ロマーニュのボーモンで生まれた．彼の父親は皮革職人で，母親は公職にある法律家の家系の出だった．そして法律家の伝統は彼女の息子が継ぐことになる．通常そうであるように，地方の学校へ通った後，ピエルはトゥールーズで勉強を続け，そこで弁護士の資格を得た．

1631 年に彼は母親のいとこと結婚した．（彼女は３人の息子と２人の娘を生む）1648 年にかれはトゥールーズの高等法院参与の地位に昇進し，1665 年に 65 歳で死ぬまでその地位を平穏かつ実直にまっとうしたのである．

フェルマーは法律家および行政家として相当の能力を持っていたが，また，主要なヨーロッパ言語や文学に関して広範な知識を持つ優れた古典学者でもあり，言語学者でもあった．

彼がラテン語，フランス語，スペイン語で作った詩は賞賛されている．しかしフェルマーは余暇の大半を何よりもお気に入りの娯楽である数学に費やした．彼は最高に理性的なものへの知的情熱に，その対象への純然たる愛情に身をこがしたのだ．

フェルマーは生涯を通じて勤勉で有能な官吏だった．では，彼は一体どのようにしてそのような一流の数学的成果を産み出す時間と精力を見出したのだろうか．その答えは（彼の日常の習慣と穏やかな気質を別にすれば），高等法院参与は地域社会からいくぶん離れた立場を取り，その社会的活動の多くを控えるように求められていたことにあるにちがいない．

フェルマーは職業的数学学者ではなかったので，公表する意思があまりなかった．実際，彼の数学的著作のほとんどは彼の死後まで出版されていない．しかしいくつかは彼の生前も手稿で出回っていた．また，彼の発見の多くを、通常は要約した形で，広い範囲に及ぶ数学上の文通者への手紙の中で公表した．

イラスト4：ピエル・ド・フェルマ（1601年—1665年），
[Mary Evans Picture Library]

（数学を築いた天才たち　上　スチュアート・ホリングデールより抜粋）

フェルマーの著作

『AD LOCOS PLANOS ET SOLIDOS』

を読んでみよう．

フェルマーは『AD LOCOS PLANOS ET SOLIDOS』の序文（日本語訳）で次のように述べている。

『数人の古代の著者が**軌跡**について記していたことに疑いはない。証人パッポスは彼の 7巻目の本の最初で「アポロニウスは平面軌跡、アリストテレスは立体軌跡を記した」といっている。しかし彼らにとって、軌跡の研究はとても容易ではなかったようである。; このことはいくつかの軌跡に対して十分な一般的な報告（次で明確にするが）をしなかったという事実から言えるのである。それゆえ、私たちは今から軌跡の研究の一般的な方法が繰り広げられるようにこの科学に適当で特別な分析を施す。

最終的な方程式に二つの未知数があるとすぐに軌跡が存在しそのうちの 1 つの端点は直線か曲線を描く。直線は一種のみであるが、曲線は無限である。: 円、放物線、双曲線、楕円など。

未知数の端点が直線や円を描くときはいつでも、平面軌跡を得る。; 放物線、双曲線、楕円を描くときは立体軌跡を得る。; もしその他の曲線が現れたときは我々は線形軌跡という。私たちはこの最後の場合には何も付け加えないだろう。なぜなら、線形軌跡の研究は推論によって平面幾何や立体幾何から簡単に得ることが出来る からである。

二つの未知数が適当な角度や場所に応じられるときこの等式は簡単に視覚化される。以後述べる事で、そのとき未知数が二次を超えなければ軌跡は平面軌跡か立体軌跡になることが明確になるだろう。』

AD LOCOS PLANOS ET SOLIDOS

ISAGOGE [1].

De locis quamplurima scripsisse veteres, haud dubium : testis Pappus initio Libri septimi [2], qui Apollonium de locis planis, Aristæum de solidis scripsisse asseverat. Sed aut fallimur, aut non proclivis satis ipsis fuit locorum investigatio; illud auguramur ex eo quod locos quamplurimos non satis generaliter expresserunt, ut infra patebit.

Scientiam igitur hanc propriæ et peculiari analysi subjicimus, ut deinceps generalis ad locos via pateat.

Quoties in ultima æqualitate duæ quantitates ignotæ reperiuntur, fit locus loco et terminus alterius ex illis describit lineam rectam aut curvam. Linea recta unica et simplex est, curvæ infinitæ : circulus, parabole, hyperbole, ellipsis, etc.

Quoties quantitatis ignotæ terminus localis describit lineam rectam aut circulum, fit locus planus; at quando describit parabolen, hyperbolen vel ellipsin, fit locus solidus; si alias curvas, dicitur locus

(1) Le texte de cet important Traité est très défiguré dans l'édition des *Varia Opera* de 1679, en particulier par l'adoption de la notation cartésienne des exposants. L'*Isagoge*, qui renferme les éléments de la Géométrie analytique moderne, et notamment une discussion de l'équation générale du second degré à deux inconnues, a cependant été rédigée et même, d'après l'article du *Journal des Savants* du 9 février 1665, communiquée par Fermat avant l'apparition de la *Géométrie* de Descartes. D'un autre côté, il est aisé de se convaincre que Fermat est toujours resté fidèle aux errements de Viète et n'a jamais fait usage dans ses écrits de la notation des exposants, sauf pour des cas exceptionnels, comme lorsqu'il faisait allusion aux travaux de Descartes.

L'existence, dans le portefeuille 1848 I de la collection Ashburnham, d'une ancienne copie de l'*Isagoge* a permis de rétablir en toute sûreté la notation employée par Fermat et d'éliminer certaines additions faites à son texte sur le manuscrit qui avait servi pour l'édition des *Varia*.

(2) Pappus, éd. Hultsch, page 636, lignes 22 et 23.

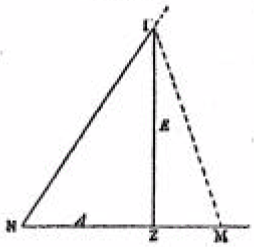
3

linearis. De hoc nihil adjungemus, quia facillime ex planorum et solidorum investigatione linearis loci cognitio derivabitur, mediantibus reductionibus.

Commode autem institui possunt æquationes, si duas quantitates ignotas ad datum angulum constituamus (quem ut plurimum rectum sumemus), et alterius ex illis positione datæ terminus unus sit datus; modò neutra quantitatum ignotarum quadratum prætergrediatur, locus erit planus aut solidus, ut ex dicendis clarum fiet.

Recta data positione sit NZM (*fig.* 78), cujus punctum datum N; NZ

Fig. 78.



æquetur quantitati ignotæ A, et ad angulum datum NZI elevata recta ZI sit æqualis alteri quantitati ignotæ E.

$$D \text{ in } A \quad \text{æquetur} \quad B \text{ in } E:$$

punctum I erit ad *lineam rectam* positione datam.

Erit enim

$$\text{ut } B \text{ ad } D, \quad \text{ita } A \text{ ad } E.$$

Ergo ratio A ad E data est, et datur angulus ad Z, triangulum igitur NIZ specie, et angulus INZ; datur autem punctum N et recta NZ positione : ergo dabitur NI positione, et est facilis compositio.

Ad hanc æqualitatem reducentur omnes, quarum homogenea partim sunt data, partim ignotis A et E admixta, vel in datas ductis vel simpliciter sumptis.

$$Z pl. - D \text{ in } A \quad \text{æquetur} \quad B \text{ in } E.$$

4

Fiat

$$D \text{ in } R \quad \text{æquale} \quad Z\,pl.;$$

erit

$$\text{ut } B \text{ ad } D, \quad \text{ita } R - A \text{ ad } E.$$

Fiat MN æqualis R : dabitur punctum M, ideoque MZ æqdabitur $R - A$. Dabitur ergo ratio MZ ad ZI; sed datur angulus ad Z, ergo triangulum IZM specie, et concludetur rectam MI junctam dari positione, ideoque punctum I erit ad rectam positione datam. Idemque nullo negotio concludetur in qualibet æqualitate cujus homogenea quædam afficientur ab A vel E.

Et est simplex hæc et prima locorum æqualitas, cujus beneficio invenientur loci omnes ad lineam rectam : verbi gratia, septima propositio Libri I *Apollonii de locis planis* ([1]), quæ generalius jam poterit enuntiari et construi.

Huic æqualitati subest pulcherrima propositio sequens, quam nos illius ope deteximus :

Si sint quotcunque rectæ lineæ positione datæ atque ad ipsas a quodam puncto ducantur rectæ in datis angulis, sit autem quod sub ductis et datis efficitur dato spatio æquale, punctum rectam lineam positione datam continget.

Infinitas omittimus, quæ Apollonianis merito possent opponi.

Secundus hujusmodi æqualitatum gradus est, quando

$$A \text{ in } E \quad \text{æq.} \quad Z\,pl.,$$

quo casu punctum I est ad *hyperbolen*.

Fiat NR (*fig.* 79) parallela ZI ; sumatur in NZ quodlibet punctum, ut M, a quo ducatur MO parallela ZI ; et fiat rectangulum NMO æquale $Z\,pl.$

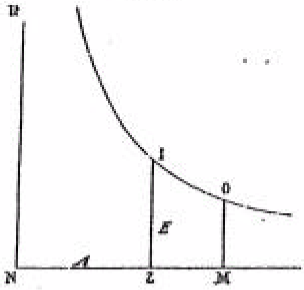
Per punctum O, circa asymptotos NR, NM, describatur hyperbole :

([1]) *Voir* plus haut, page 24, note 1.

5

dabitur positione et transibit per punctum I, quum ponatur rectangulum A in E, sive NZI, æquale NMO.

Fig. 79.



Ad hanc æqualitatem reducentur omnes quarum homogenea partim sunt data, vel ab A aut E aut A in E affecta.

Ponatur

$$D pl. + A \text{ in } E \quad \text{æq.} \quad R \text{ in } A + S \text{ in } E.$$

Igitur, ex artis præceptis,

$$R \text{ in } A + S \text{ in } E - A \text{ in } E \quad \text{æquabitur} \quad D pl.$$

Effingatur rectangulum abs duobus lateribus, in quo homogenea

$$R \text{ in } A + S \text{ in } E - A \text{ in } E$$

reperiantur : erunt duo latera

$$A - S \quad \text{et} \quad R - E$$

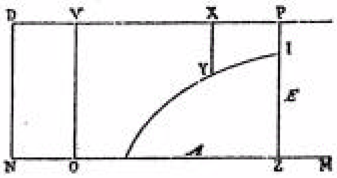
et rectangulum sub ipsis æquabitur R in $A + S$ in $E - A$ in $E - R$ in S.

Si igitur a $D pl.$ abstuleris R in S, rectangulum sub $\overline{A - S}$ in $\overline{R - E}$ æquabitur $D pl. - R$ in S.

Fiat NO (*fig.* 80) æqualis S, et ND, parallela ZI, fiat æqualis R; per punctum D ducatur DP parallela NM, < per punctum O > OV parallela ND, et ZI producatur in P.

Quum NO æquetur S, et NZ, A, ergo $A - S$ æquabitur OZ sive VP; similiter, quum ND, sive ZP, æquetur R, et ZI, E, ergo $R - E$ æquabitur PI : rectangulum igitur sub VP in PI æquatur dato *Dpl.* — *R* in *S*. Ergo punctum I erit ad hyperbolen, cujus asymptoti PV, VO.

Fig. 80.



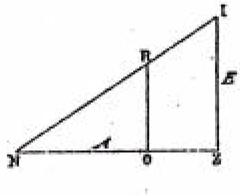
Rectangulo enim *Dpl.* — *R* in *S* æquetur, sumpto quovis puncto X et ductâ parallelâ XY, rectangulum VXY, et per punctum Y, circa asymptotos PV, VO, hyperbole describatur : per punctum I transibit, nec est difficilis in quibuslibet casibus analysis aut constructio.

Sequens æqualitatum localium gradus est, quum *Aq.* vel æquatur *Eq.*, vel est in ratione data ad *Eq.*, vel etiam *Aq.* + *A* in *E* est ad *Eq.* in data ratione; denique hic casus omnes æquationes comprehendit intra metam quadratorum, quarum homogenea omnia vel a quadrato *A*, vel a quadrato *E*, vel a rectangulo *A* in *E* afficiuntur.

His omnibus casibus punctum I est ad *lineam rectam*, cujus rei demonstratio facillima.

Sit NZ quad. + NZ in ZI ad ZI quad. in ratione data (*fig.* 81).

Fig. 81.



Ducatur quævis parallela OR; quadratum NO + NO in OR erit ad OR quadratum in eadem ratione, ut est facillimum demonstrare. Punctum igitur I erit ad rectam positione < datam >.

[Sumatur enim quodvis punctum, ut O, et fiat data ratio quadrati

NO + NO in OR ad OR quadratum. Juncta NR dabitur positione et satisfaciet proposito] (¹), idemque continget in quibuslibet æquationibus, quarum omnia homogenea a potestatibus ignotarum vel rectangulo sub ipsis afficientur, ut inutile sit singulos casus scrupulosius percurrere.

Si potestatibus ignotarum vel rectangulis sub ipsis admisceantur homogenea, partim omnino data, partim sub data recta in alteram ignotarum, difficilior evadet constructio : singulos casus construimus breviter et demonstramus.

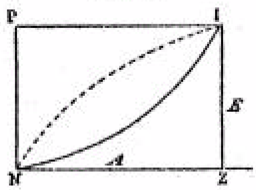
Si

$$A q. \quad \text{æquatur} \quad D \text{ in } E,$$

punctum I est ad *parabolen*.

Fiat NP parallela ZI (*fig.* 82), et circa diametrum NP describatur

Fig. 82.



parabole, cujus rectum latus recta D data, et applicatæ sint parallelæ NZ : punctum I erit ad parabolen hanc positione datam.

Ex constructione rectangulum sub D in NP æquabitur quadrato PI, hoc est, rectangulum sub D in IZ æquabitur quadrato NZ, ideoque :

$$D \text{ in } E \quad \text{æquabitur} \quad A q.$$

Ad hanc æquationem facillime reducentur omnes in quibus $Aq.$ miscetur homogeneis sub datis in E, aut $Eq.$ homogeneis sub datis in A,

(¹) La démonstration mise entre crochets est suspecte à divers titres; si elle n'a pas été interpolée, on ne peut guère la considérer que comme un reste d'une première rédaction de Fermat.

idemque continget, licet homogenea omnino data æquationibus misceantur.

Sit

$$Eq. \quad \text{æquale} \quad D \text{ in } A.$$

In præcedenti figura, vertice N, circa diametrum NZ, describatur parabole, cujus rectum latus sit D, et applicatæ rectæ NP parallelæ : præstabit propositum, ut patet.

Ponatur

$$Bq. - Aq. \quad \text{æq.} \quad D \text{ in } E.$$

Ergo

$$Bq. - D \text{ in } E \quad \text{æquabitur} \quad Aq.$$

Applicetur $Bq.$ ad D et sit æquale D in R. Ergo

$$D \text{ in } R - D \text{ in } E \quad \text{æquabitur} \quad Aq.,$$

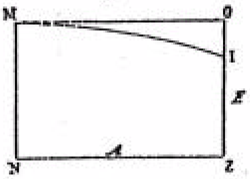
ideoque

$$D \text{ in } (R - E) \quad \text{æquabitur} \quad Aq.$$

Ideoque hæc æquatio reducetur ad præcedentem : recta quippe $R - E$ succedet ipsi E.

Fiat quippe (*fig.* 83) NM parallela ZI et æqualis R, et per punctum M ducatur MO parallela NZ : datur punctum M, et recta MO positione. In

Fig. 83.



hac constructione, OI æquatur $R - E$: ergo D in OI æquatur NZ quad., sive MO quad. Vertice M, circa diametrum MN, descripta parabole, cujus rectum latus D, et applicatæ ipsi NZ parallelæ, præstabit propositum, ut patet ex constructione.

Si

$$Bq. + Aq. \quad \text{æq.} \quad D \text{ in } E,$$

$$D \text{ in } E - Bq. \quad \text{æquabitur} \quad Aq.,$$

etc. ut supra. Similiter omnes æquationes ab E et $Aq.$ affectæ construentur.

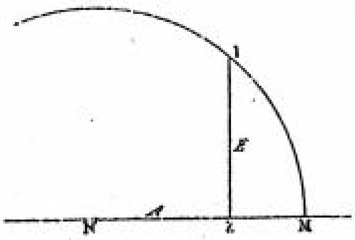
Sed $Aq.$ miscetur sæpe $Eq.$ et homogeneis omnino datis.

$$Bq. - Aq. \quad \text{æquetur} \quad Eq. :$$

punctum I est ad *circulum* positione datum, quando angulus NZI est rectus.

Fiat NM (*fig.* 84) æqualis B; circulus centro N, intervallo NM, descriptus præstabit propositum, hoc est : quodcumque punctum sumpseris in ipsius circumferentia, ut I, quadratum ZI æquabitur quadrato NM (sive $Bq.$) — quadrato NZ (sive $Aq.$), ut patet.

Fig. 84.



Ad hanc æquationem reducentur omnes affectæ ab $Aq.$ et $Eq.$, et ab A vel E in datas ductis, modò angulus NZI sit rectus, et prætereca coefficientes $Aq.$ æquentur coefficientibus $Eq.$

Sit

$$Bq. - D \text{ in } A \text{ bis} - Aq. \quad \text{æquale} \quad Eq. + R \text{ in } E \text{ bis}.$$

Addatur utrimque $Rq.$, ut $E + R$ succedat E : fiet

$$Rq. + Bq. - D \text{ in } A \text{ bis} - Aq. \quad \text{æquale} \quad Eq. + Rq. + R \text{ in } E \text{ bis}.$$

Ipsis $Rq.$ et $Bq.$ addatur $Dq.$, ut $D + A$ succedat ipsi A, et summa quadratorum $Rq.$, $Bq.$, et $Dq.$ æquetur $Pq.$ Ergo

$$Pq. - Dq. - D \text{ in } A \text{ bis} - Aq. \quad \text{æquabitur} \quad Rq. + Bq. - D \text{ in } A \text{ bis} - Aq.;$$

nam ek constructione

$$Pq. - Dq. \quad \text{æquatur} \quad Rq. + Bq.$$

10

Si igitur loco ipsius $A + D$ sumpseris A et loco $E + R$ sumpseris E, fiet

$$Pq. - Aq. \quad \text{æquale} \quad Eq.,$$

et reducetur æquatio ad præcedentem.

Simili ratiocinatione similes æquationes reducentur, et hac via omnes propositiones secundi Libri *Apollonii De locis planis* (¹) construximus, et sex priores in quibuslibet punctis habere locum demonstravimus : quod sane mirabile est et ab Apollonio fortasse ignorabatur.

Sed

$$Bq. - Aq. \text{ ad } Eq. \text{ habeat rationem datam,}$$

punctum I erit ad *ellipsin*.

Fiat MN æqualis B, et per verticem M, diametrum NM, centrum N, describatur ellipsis, cujus applicatæ sint rectæ ZI parallelæ et quadrata applicatarum ad rectangulum sub segmentis diametri habeant rationem datam : punctum I erit ad hujusmodi ellipsin. Etenim quadratum NM — quadrato NZ æquatur rectangulo sub diametri segmentis.

Ad hanc reducentur similes in quibus $Aq.$ ex una parte opponetur $Eq.$ sub contraria affectionis nota et sub coefficientibus diversis. Nam si coefficientes sint eædem et angulus sit rectus, locus erit ad circulum, ut jam diximus; licet igitur coefficientes sint eædem, modò angulus non sit rectus, locus erit ad ellipsin, et, licet immisceantur æquationibus homogenea sub datis et A vel E, fiet reductio eo quod jam usurpavimus artificio.

Si

$$Aq. + Bq. \text{ est ad } Eq. \text{ in data ratione,}$$

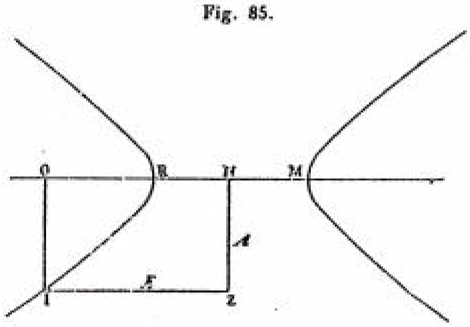
punctum I est ad *hyperbolen*.

Fiat NO (*fig.* 85) parallela ZI; data ratio sit eadem quæ $Bq.$ ad quadratum NR : dabitur ergo punctum R. Circa diametrum RO, per ver-

(¹) *Voir* plus haut, pages 29 et 30, note 2.

ticem R, centrum N, describatur hyperbole, cujus applicatæ sint parallelæ NZ, et rectangulum sub toto diametro et RO unà cum RO quadrato ad quadratum OI sit in data ratione, NR quadrati ad *Bq.* Ergo, componendo, rectangulum sub MOR (positâ MN æquali NR) unà cum quadrato NR erit ad quadratum OI unà cum *Bq.* in ratione data, NR quadrati ad *Bq.* Sed rectangulum MOR, unà cum NR quadrato, æqua-

Fig. 85.



tur NO quadrato, sive ZI quadrato, sive *Eq.*; et quadratum OI unà cum *Bq.* æquatur quadrato NZ (sive *Aq.*) unà cum *Bq.* : ergo est

ut *Eq.* ad *Bq.* + *Aq.*, ita NR quad. ad *Bq.*

et, convertendo,

Bq. + *Aq.* est ad *Eq.* in ratione data.

Punctum igitur I est ad hyperbolen positione datam.

Eodem quo jam usi sumus artificio, ad hanc æqualitatem reducentur omnes quæ ab *Aq.* et *Eq.* afficiuntur unà cum datis, sive simpliciter, sive misceantur ipsis homogenea sub *A* vel *E* in datas, modò *Aq.* habeat eamdem ex altera parte affectionis notam, quam *Eq.* Nam, si sint diversæ, propositio concludetur per circulos vel ellipses.

Difficillima omnium æqualitatum est quando ita misceantur *Aq.* et *Eq.* ut nihilominus homogenea quædam ab *A* in *E* afficiantur unà cum datis, etc.

Bq. — *Aq.* bis æquetur *A* in *E* bis + *Eq.*

12

Addatur utrimque $Aq.$, ut $A+E$ sit latus alterius ex homogeneis : ergo

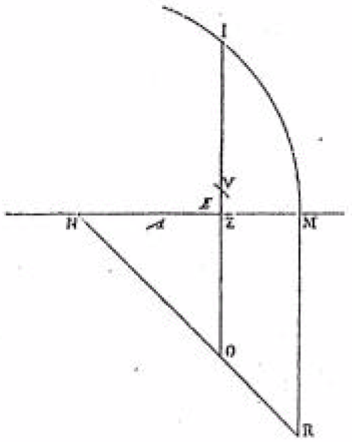
$$Bq.-Aq. \text{ æquabitur } Aq.+Eq+A \text{ in } E \text{ bis.}$$

Pro $A+E$ sumatur E, si placet, et ex præcedentibus circulus MI (*fig.* 86) præstet propositum, hoc est :

MN quad. (sive $Bq.$) — NZ quad. (sive $Aq.$)
æquetur quadrato ZI (sive quadrato abs $\overline{A+E}$).

Fiat VI æqualis NZ, sive A : ergo ZV æquatur E. In hac autem quæstione punctum V, sive extremum rectæ E, tantum inquirimus : videndum ergo et demonstrandum ad quam lineam sit punctum V.

Fig. 86.



Fiat MR parallela ZI et æqualis MN, et jungatur NR, ad quam producta IZ incidat ad punctum O. Quum MN æquetur MR, ergo NZ æquabitur ZO; sed NZ æquatur VI : ergo tota VO toti ZI est æqualis, ideoque

quadratum MN — quadrato NZ æquatur quadrato VO.

Datur autem triangulum NMR specie : ergo quadrati NM ad quadratum NR datur ratio, ideoque et quadrati NZ ad quadratum NO dabitur ratio. Ratio igitur

quadrati MN — quadrato NZ ad quadratum NR — quadrato NO datur; probavimus autem

quadratum OV æquari quadrato MN — quadrato NZ :

ergo ratio quadrati NR — NO quadrato ad quadratum OV datur. Dantur autem puncta N et R, et angulus NOZ : ergo punctum V, ex superius demonstratis, est ad ellipsin.

Non absimili methodo ad superiores casus reducentur reliqui, in quibus homogenea sub A in E homogeneis partim datis, partim sub $Aq.$ aut $Eq.$ immiscebuntur, aut etiam sub A et E in datas ductis, cujus rei disquisitio facillima ; semper enim beneficio trianguli specie noti construetur quæstio.

Breviter igitur et dilucide complexi sumus quidquid de locis planis et solidis inexplicatum veteres reliquere, constabitque deinceps ad quem locum pertinebunt casus omnes propositionis ultimæ Libri I *Apollonii de locis planis* (¹), et omnia omnino ad hanc materiam spectantia nullo negotio detegentur.

Sed libet coronidis loco pulcherrimam hanc propositionem adjungere, cujus facilitas statim innotescet.

Si, positione datis quotcumque lineis, ab uno et eodem puncto ad singulas ducantur rectæ in datis angulis, et sint species ab omnibus ductis dato spatio æquales, punctum contingit positione datum solidum locum.

Unico exemplo fit via ad practicen : Datis duobus punctis N, M (*fig.* 87), inveniendus locus a quo si jungas rectas IN, IM, quadrata IN, IM ad triangulum INM datam habeant rationem.

Recta NM æquetur B, et recta ZI, ad angulos rectos, dicatur E terminus; NZ dicatur A : ergo, ex artis præceptis,

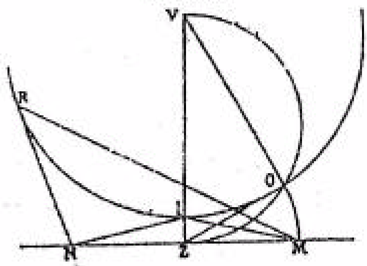
$$Aq.\text{bis} + Bq. - B \text{ in } A \text{ bis} + Eq.\text{bis} \quad \text{ad rectangulum } B \text{ in } E$$

habebit rationem datam et, resolvendo hypostases ex jam traditis præceptis, ita procedet constructio :

(¹) *Voir* plus haut, p. 27, la note sur le sens qu'il faut attribuer à cette proposition d'Apollonius.

NM bifariam secetur in Z; a puncto Z excitetur perpendicularis ZV, et fiat data ratio eadem quæ ZV quadruplæ ad NM; descripto semicirculo VOZ super VZ (¹) applicetur ZO æqualis ipsi ZM, et junctâ VO, centro V, intervallo VO, describatur circulus OIR, in quo sumatur

Fig. 87.



quodlibet punctum, ut R, et jungantur rectæ RN, RM : Aio quadrata RN, RM ad triangulum RNM esse in data ratione.

Hæc inventio, si libros duos *de locis planis* a nobis dudum restitutos præcessisset, elegantiores sane evasissent localium theorematum constructiones : nec tamen præcocis licet et immaturi partûs nos adhuc pœnitet, et informes ingenii fœtus posteris non invidere scientiæ ipsius quadamtenus interest, cujus opera primo rudia et simplicia novis inventis et roborantur et augescunt. Imo et studiosorum interest latentes ingenii progressus et artem sese ipsam promoventem penitus habere perspectam.

There is no doubt that several ancient authors have written on loci, witness Pappus, who, at the beginning of his seventh book, states that Apollonius had written on plane loci and Aristaeus on solid loci.[4] But it seems that to them the study of loci did not come quite easily; this we gather from the fact that for several loci they did not give a sufficiently general account, as will be clarified by what follows here.

We shall therefore submit this science to an appropriate and particular analysis, so that from now on a general way to the study of loci shall be opened.

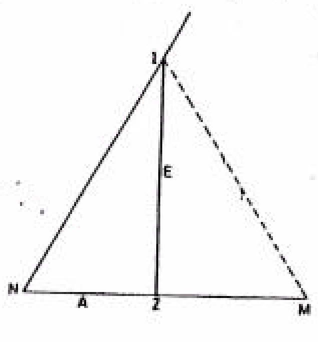
As soon as in a final equation [*aequalitas*] two unknown quantities appear, there exists a locus, and the end point of one of the two quantities describes a straight or a curved line. The straight line is the only one of its kind, but the types of curves are infinite: a circle, a parabola, a hyperbola, an ellipse, etc.

Whenever the end point of the unknown quantity describes a straight line or a circle, we have a plane locus; when it describes a parabola, hyperbola, or ellipse, then we have a solid locus; if other curves appear, then we say that the locus is a linear locus [*locus linearis*]. We shall not add anything to this last case, since the study of the linear locus can easily be derived from that of plane and solid ones by means of reductions.

The equations can be easily visualized [*institui*], when the two unknown quantities are made to form a given angle, which we usually take to be a right one, with the position and the end point of one of them given. Then, if no one of the unknown quantities exceeds a square, the locus will be plane or solid, as will be clear from what we shall say.

Let NZM be a straight line given in position, N a fixed point [Fig. 2] on it. Let NZ be one unknown quantity A, and the segment ZI, applied to it at given angle NZI, be equal to the other unknown quantity E. When D times A is equal to B times E, the point I will describe a straight line given in position, since B is to D as A is to E.[5] Hence the ratio of A to E is given, and, since the angle at Z is given, the form of the triangle NIZ, and with it the angle INZ, is given. But the point N is given and the straight line NZ is given in position: hence NI is given in position and it is easy to make the synthesis [*compositio*].

Fig. 2



To this equation all equations can be reduced of which the terms [*homogenea*] are partly given, partly mixed with the unknowns A and E, either multiplied with the given quantities, or appearing simply. Let Z pl $- D$ times A equal B times E. Let D times R be Z pl. Then we will find that B is to D as $R - A$ is to E. Let us take MN equal to R, then point M is given, hence MZ is equal to $R - A$. Hence the ratio of MZ to ZI is known, but the angle at Z is given, hence also the form of the triangle IZM. We conclude that the straight line MI is given in position. Thus point I will be on a straight line given in position.[6] We reach the same result without difficulty for any equation containing the terms A and E.

This is the first and simplest equation of a locus, by means of which all the loci dealing with a straight line can be found; see, for example, the seventh proposition of Book I of Apollonius *On plane loci*, which has since found a more general formulation and construction. This equation also leads to the following elegant proposition, which we discovered with its help:

Let any number of straight lines be given in position. From some point draw to them straight lines at given angles. If the sum of the products of the lines thus drawn with the given lines is equal to a given area, then the point will be on a straight line given in position.[7]

We omit a great number of other propositions, which could be considered as corollaries to those of Apollonius.

The second species of equations of this kind are of the form

$$A \text{ times } E \text{ is } Z \text{ pl},^{8}$$

in which case point I traces a *hyperbola*. Draw NR parallel to ZI; through any point, such as M, on the line NZ, draw MO parallel to ZI. Construct the rectangle NMO equal in area to Z pl. Through the point O, between the asymptotes NR, NM, describe a hyperbola; its position is determined and it will pass through point I, since we have assumed, as it were, AE—that is to say, the rectangle NZI—equivalent to the rectangle NMO [Fig. 4].

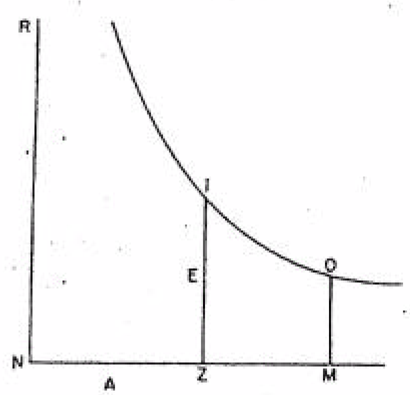


Fig. 4

To this equation we may reduce all those whose terms are in part constant, or in part contain A or E or AE.

If we let

$$D \text{ pl} + A \text{ times } E \text{ equal } R \text{ times } A + S \text{ times } E,$$

then we obtain, by well-known methods,

$$R \text{ times } A + S \text{ times } E - A \text{ times } E \text{ equal } D \text{ pl}.$$

Let us construct a rectangle on two sides such that the terms R times $A + S$ times $E - A$ times E are contained in it; then the two sides will be $A - S$ and $R - E$ and the rectangle on them will be equal to R times $A + S$ times $E - A$ times $E - R$ times E.

If now we subtract R times S from D pl, then the rectangle on $A - S$ and $R - E$ will be equal to D pl $-$ R times S.

Take NO equal to S and ND, parallel to ZI, equal to R. Through point D, draw DP parallel to NZ; through point O, draw OV parallel to ND; prolong ZI to P [Fig. 5].

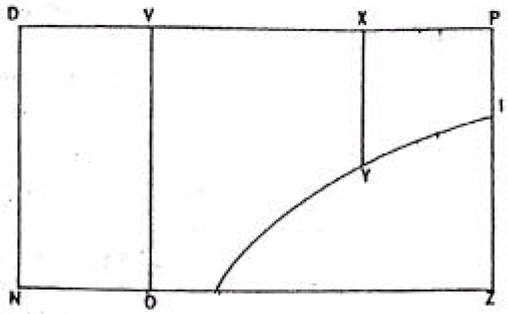


Fig. 5

Since $NO = S$ and $NZ = A$, we have $A - S = OZ = VP$. Similarly, since $ND = ZP = R$ and $ZI = E$, we have $R - E = PI$. The rectangle on PV and PI is therefore equal to the given area D pl $-$ R times E. The point I is therefore on a hyperbola having PV, VO as asymptotes.[9]

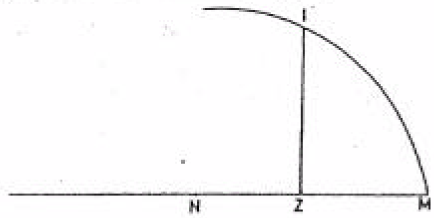
If we take any point X, the parallel XY, and construct the rectangle VXY, and through point Y we describe a hyperbola between the asymptotes PV, VO, it will pass through point I. The analysis and construction are easy in every case.

The next species of equations for loci arises if we have A^2 equal to E^2, or in given ratio to E^2, or, again if $A^2 + A$ times E is in given ratio to E^2. Finally this type includes all the equations whose terms are of the second degree containing either A^2, E^2, or the rectangle on A and E. In all these cases the point I traces a *straight line*, as can easily be demonstrated.[10]

But if $Bq - Aq$ is to Eq in a given ratio, then the point I will be on an *ellipse*.

Let MN be equal to B, and let an ellipse be described with M as vertex, NM as diameter, and N as center, of which the ordinates [*applicatae*] are parallel to the straight line ZI. The squares of the ordinates must have a given ratio to the rectangle formed by the segments of the diameter. Then the point I will be on this ellipse. Indeed, the square on NM − the square on NZ is equal to the rectangle formed by the segments of the diameter [Fig. 6].[11]

Fig. 6

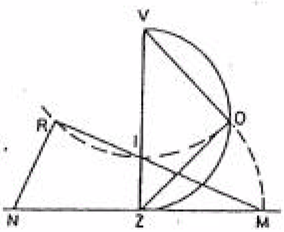


To this equation can be reduced all those in which Aq is on one side of the equation and Eq with an opposite sign and a different coefficient on the other side. If the coefficients are the same and the angle a right angle, the locus will be a circle, as we have said. If the coefficients are the same, but the angle is not a right angle, the locus will be an ellipse.

Moreover, though the equations include terms which are products of A or E by given magnitudes, the reduction may nevertheless be made by the artifice which we have already employed.

A single example will suffice to indicate the general method of construction. Given two points N and M, required the locus of the points such that the sum of the squares of IN, IM shall be in a given ratio to the triangle INM [Fig. 7].

Fig. 7



Let $NM = B$, let E be the line ZI drawn at right angles to NM, and let A be the distance NZ. According to well-known methods we find that A^2 bis + B^2 − B times A bis + E^2 bis is to rectangle B times E in a given ratio.[12] Following in treatment the procedures previously explained we have the suggested construction.

19

Bisect NM at Z; erect at Z the perpendicular ZV; make the ratio $4ZV$ to NM equal to the given ratio. On VZ draw the semicircle VOZ, inscribe ZO equal to ZM, and draw VO. With V as center and VO as radius draw the circle OIR. If from any point R on this circle, we draw RN, RM, I say that the sum of the squares of RN and RM is in the given ratio to the triangle RNM.

The constructions of the theorems on loci could have been much more elegantly presented if this discovery had preceded our previous revision of the two books *On plane loci*. Yet, we do not regret this work, however precocious or insufficiently ripe it may be. In fact, there is for science a certain fascination in not denying to posterity works that are as yet spiritually incomplete; the labor of the work, at first simple and clumsy, gains strength as well as stature through new inventions. It is quite important that the student should be able to discern clearly the progress which appears veiled as well as the spontaneous development of the science.

ワークシート①

ギリシア文明（ＢＣ６００～ＡＤ６００ごろ）における数学は今なお高く評価されています．当時ギリシア人は記号代数を持っていませんでした．したがって，あらゆることを幾何的に考えなければなりませんでした．例えば，数を「線分」，二つの数の積を「面積」として考えていた時代です．そんな文化です．

★ 様々な式（記号代数）の幾何的表現をしてみよう．

「a」の幾何的表現

「b」の幾何的表現

とする．このとき，

（1）「a ＋ b」の幾何的表現

（2）「a × b」の幾何的表現

ワークシート②

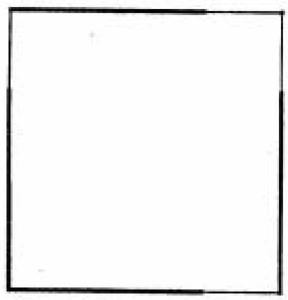
（3）「a^2」の幾何的表現

（4）「$(a + b)^2$」の幾何的表現

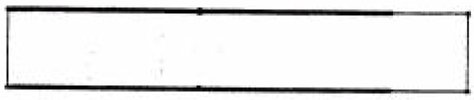
ワークシート③

★ 次の幾何的表現を式（記号代数）で表してみよう.

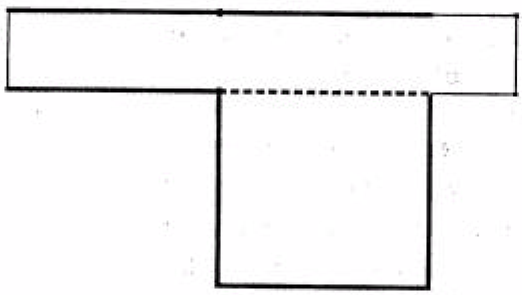
（1）



（2）



（3）



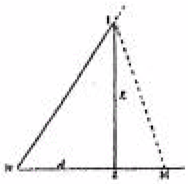
【MEMO】

ワークシート④

『AD LOCOS PRANOS ET SOLIDOS』

（英語訳 Struik,A Source Book in Math）

Let *NZM* be a straight line given in position , *N* a fixed point on it .①



Let *NZ* be one unknown quantity *A*② ,

and the segment *ZI* , applied to it at given angle *NZI* , ③

be equal to the other unknown quantity *E*. ③

When *D* times *A* is equal to *B* times *E* , ④

the point *I* will describe a straight line given in position .⑤

since *B* is to *D* as *A* is to *E* .⑥

Hence the ratio of *A* to *E* is given⑦ ,and , since the angle at *Z* is given , the form of the triangle *NIZ* ,and with it the angle *INZ* is given⑧ . But the point *N* is given and the straight line *NZ* is given in position⑨; hence *NI* is given in position and it is easy to make the synthesis⑩

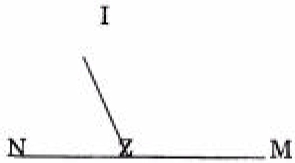
ワークシート⑤

フェルマーの『AD LOCOS PLANOS ET SOLIDOS』によれば

直線 NZM を与えて，
NZ の長さを変数（　　　　），ZI の長さを変数（　　　　）とし，
それらの関係に

D×A＝B×E　つまり（　　　　）：（　　　　）＝B：D　①
を与えるとき，
点 I は（　　　　）を描く．

ここで∠（　　　　）には任意の角度を与えている。　②



フェルマーは①，②つまり A と E に関する方程式と∠NZI を与えると直線が描けることを発見した。

演習　D＝3E（B＝1，D＝3）で次の角が与えられる時の点 I の軌跡を書け．

（1）∠NZI＝30°

（2）∠NZI＝90°

（3）∠NZI＝120°

ワークシート⑥

『AD　LOCOS　PRANOS　ET　SOLIDOS』

（英語訳　Struik,A Source Book in Math）

The secound species of equations of this kind are of the form

A times E is Zpl ①

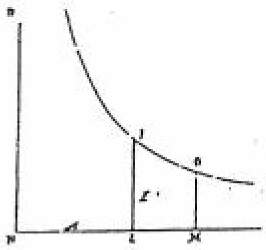
in which case point I traces a hyperbole.②

Draw NR parallel to ZI; through any point , such as M, on the line NZ ③,

draw MO parallel to ZI③. Construct the rectangle NMO equal in area to Zpl④

Through the point O, between the asymptotes NR, NM, describe a hyperbole⑤ :

its position is determined and it will pass through point I⑥ , since we have assumed , as it were , AE －that is to say , the rectangle NZI－equivalent to the rectangle NMO⑦



ワークシート⑦

『AD LOCOS PRANOS ET SOLIDOS』を読んで，理解してみよう。

$Zpl = Z^2$　trace ～をたどる　hyperbole 双曲線　parallel 平行な　rectangle 長方形　area 面積
construct 作図する　asymptotes 漸近線　assume 仮定する　as it were いわば　equivalent 等しい