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Abstract

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THE DEVELOPMENT OF LANGUAGE ABOUT FUNCTION: AN APPLICATION OF VAN HIELE'S LEVELS

Masami Isoda
Institute of Education, The University of Tsukuba

ABSTRACT

This paper proposes a model of the development of language about function. This model was developed by comparing Japanese teaching practices and national curriculum with generalized forms of van Hiele's Levels. This paper points out features of van Hiele's Levels and shows that they are also characteristics of the proposed levels of language about functions. These features include: language hierarchy, the existence of un-translatable concepts, a duality of object and method, and mathematical language and student thinking in context. The levels indicate that students' development resembles an expanding equilibration, rather than a monotonous increase of knowledge.

Introduction

In the past ten years, multi-representational tools for exploring functions have been changing the contexts and learning sequence of arithmetic, pre-algebra, algebra, pre-calculus and calculus. In discussing these reforms, it is important to consider students' development not only in terms of conceptual functional thinking but also considering students' language concerning functions. Several models of the development of functional thinking have been proposed (E. Dubinsky, 1992; A. Sfard, 1991; A. Sierpinska, 1992; S. Vinner, 1991). These models' views imply that the development of students' knowledge and thinking about function is like an expanding equilibration rather than a monotonous increase (cf. J. Confrey 1994; E.V. Glasersfeld 1995). This paper will show another model of development which provides the same view but focuses on the students' development concerning the representations of function as mathematical language. One characteristic of this model is its background. This model was set by comparing the Japanese national curriculum and teaching practices with the generalized forms of van Hiele's Levels (A. Hoffer 1983; M. Isoda 1984). The Japanese curriculum may be the only national curriculum which has specified areas of function/functional thinking from elementary school.[1] This paper discusses the features of this model from the view point of expanding equilibration and the features of van Hiele's Levels.

[1] In the national curriculum, the 4 areas of elementary school and 5 areas of junior high school mathematics have been formally in place since 1958. These areas include functional thinking, figures/geometry, and arithmetic/algebra. Each area is connected and integrated with each other. This tradition has its roots in the movement of Perry, Kline & Moore.

The Features of van Hiele Levels

A. Language Hierarchy. Each level has its own language and the levels are hierarchical (van Hiele, 1959).

B. Existence of Un-translatable Concepts. The corresponding contents of different levels sometimes conflict (van Hiele, 1986).

C. Duality of Object and Method. The thinking of each level has its own inquiring object (subject matter) and inquiring method (the way of learning). The method of each level is verbalized and becomes the object, subject matter, of the next level's inquiring. This is the duality between object and method (van Hiele 1958; H. Freudenthal 1973; I. Hirabayashi 1978).

D. Mathematical Language and Student Thinking in Context. While the levels are distinguished as sets of mathematical language, the actual thinking of each student varies depending on the teaching and learning context (van Hiele, 1958; M. Isoda, 1988; D.Clements, 1992; cf. M. Battista, 1994).

The last feature claims that we should make a distinction between the levels of mathematical language and the levels of students thinking itself. Although several research studies have attempted to evaluate individual student's levels of geometric thinking, they point out the difficulty in doing so (cf. J. Mayberry, 1983; A. Gutiérrez, 1991; D.Clements, 1992). If these four features can be pointed out in another area of mathematics, for example the area termed 'functional relation' in Japan, we could conclude that it is an application of van Hiele's Levels. This paper first discusses the levels of function from the viewpoint of language and then discusses the development of students' skills.

The Levels of Language about Functions

Through investigations[2] of the development of students' language for describing functions and the history of the description of motion, the following levels of function have been discussed[3,4] (M. Isoda, 1987, 1988, 1990). Historical examples are written in footnote five through eleven.

Level 1. Level of Everyday Language.

Students describe relations in phenomena using everyday language obscurely. They can discuss changes in numbers using calculations, but usually their descriptions are done with or focused

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[2]Investigations included tests, interviews and teaching practice/classroom observations.

[3]In van Hiele theory, levels are described with like these generalized students' activities. But these generalized description are already mentioned the level of language rather than each student's thinking itself. Because depending on the context/educational situation, students could do more higher level activity and students' activity usually included lower level activity and change depending on context.

[4]Because curriculum and students' development are mutually related, students' development reflects the curriculum and investigations of development cannot prove its hierarchy. Phylogenetic examples are a good ground for ontogenesis.

on one physically evident variable, the *dependent variable*.[5] Even if they are aware of covariation, it is difficult for them to explain it appropriately using two variables because their descriptions of relations are done obscurely[6] using everyday language . So it is difficult for them to compare different phenomena at once, appropriately.

Level 2. Level of Arithmetic

Students describe the rules of relations using *tables*. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than with the only everyday language of Level 1.[7] Students have general concepts about some rules of relations[8] , for instance, *proportion*. Students can compare different phenomena using such rules. They describe rules of relations as covariation and when reading tables, their interpretation of the covariation of variables is at least as strong as their interpretation of correspondence. Students can use formulas and graphs to represent rules and relations, too, but it is not easy for them to translate between notations.

Level 3. Level of Algebra and Geometry

Students describe functions using *equations* and *graphs*. To explore function, they translate among the notations of tables, equations and graphs and use algebra and geometry.[9] At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as the mental image. For example, they can easily find the equation emerging from the graph, and the graph from the equation.

Level 4. Level of Calculus

Students describe function using calculus. In calculus, functions are described in terms of *derived* or *primitive functions*.[10] For example, to describe the features of a function we use its derived function which is already learned. The theory of calculus is a generalized theory of this type of description.

Level 5. Level of Analysis

An example of language for description is functional analysis which is a metatheory of calculus. This level's justification is based on historical development[11] and not yet investigated.

Table 1 shows the duality between object and method in van Hiele's Levels (the Levels of Geometry) and in the Levels of Function. Examples of untranslatable concepts are offered between each level. Furthermore, the existence of

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[5]Zeno, Eleatic school, argued that Achilles could not catch a tortoise.
[6]Aristotle wrote that something falls faster if it is heavier.
[7]Ptolemy made the chord (trigonometry) table to describe the motion of planets.
[8]Galileo found the ratio of differences in the distance fallen of falling bodies to be the sequence of odd numbers.
[9]Galileo found the parabola, which Apollonius had described as being cut from a conic, from the odd number ratio of a falling body.
[10]Newton descried motion using fluxion.
[11]J. Bernoulli posed the problems of branchistochrone and geodesic line. These variational problems were origin of functional analysis and differential geometry.

duality and untranslatable concepts suggests a hierarchical relationship between the levels.[12] Thus, these constitute three of the four features of van Hiele's Levels listed earlier in the paper. These, as well as the fourth feature, will be further discussed later from the viewpoint of the development of student thinking.

Table 1. Duality and Un-translatable Concepts

	The Levels of Geometry	The Levels of Function
Level 1	Students explore matter (object) using *figures* (*method*).	Students explore phenomena (object) using obscure *relations* or *variation* (*method*).
Example of conflicts between levels	Because it has rounded corners, the road sign 'YIELD' is not a triangle according to the meanings of Level 2, but we call it a triangle in daily language.	In Japanese, we use "2 BAI, 3 BAI" to mean "two times, three times" on level 2. But in everyday Japanese (Level 1), we can use "BAI" to mean either "double" or "plus". A child on level 1 says "BAI,BAI" ("plus plus") to mean three times the original amount. But "BAI,BAI" ("double double") usually means four times. On level 2, students use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3 BAI" and do not say it "BAI,BAI".
Level 2	Students explore the figures using the *property*; The object on level 2 was the method on level 1.	Students explore the relations using *rules*; The object on level 2 was the *method* on level 1.
Example of conflicts	A square is rectangular on Level 3, but not on Level 2.	The constant function is a function on Level 3 but 'constant' is not the relation which was discussed as covariation on level 2.
Level 3	Students explore the properties of figures using *implication*.	Students explore the rules using *notations of functions*.
Example of conflicts	The isosceles triangle has congruent angles. On Level 3, it is induced already and we do not have to explain more. On Level 4, we prove it.	On Level 3, a tangent line of quadrilateral function deduced using the property of only one common point / a multiple root. On the Level 4, the tangent line does not always have this property.
Level 4	Students explore the proposition, which is formed by implication, using *proof*.	Students explore functions using *derived or primitive function*.

The Development of Students' Thinking

Students' development from a lower level to a higher level resembles an expanding equilibration rather than a monotonous increase. Below, two examples are offered which were selected from investigations of the development of functional language from level 2 to level 3. The features of van Hiele's Levels help

[12]In the case of quadratic function, we can make the following distinctions.

Level 1. Students do not easily compare the situations. They can not appropriately distinguish quadratic from other situations if we use only daily language. See footnote No. 6.

Level 2. Quadratic functions and contextual situations can be described using a table where second differences are constant.

Level 3; Quadratic functions are described algebraically by $y=ax^2+bx+c$, and geometrically by parabolas. Tangent lines are discussed using b^2-4ac.

Level 4; Quadratic function is described with the derived function of cubic function and primitive function of linear function. Tangent lines are discussed using derivative.

explain the students' growth of knowledge. First, I describe the Japanese curriculum for moving from level 2 to level 3.

In the national curriculum in Japan, an informal notion of proportion is taught in grade 4 and more formal concepts of whole number proportion including y=ax are taught via real situations in grade 6. The curriculums of grades 4 through 6 are regarded as level 2 or as a transition to level 2. In grade 7 (junior high school grade 1 in Japan), students learn how to solve equations with one variable, the definition of function using the idea of correspondence, and the function y=ax. In grade 8, the linear function y=ax+b is taught. In grade 9, the quadratic function $y=ax^2$ is taught and function is redefined using the idea of set and correspondence[13]. The curriculums of grades 7 through 9 are regarded as level 3 or as a transition to level 3. The investigation found that many students in grade 6 thought on level 2, and many in grade 10 thought on level 3.

Example 1. Students lose their connection to lower levels of thinking in the process of moving to a higher level: The results[14] of problem 1[15], below, show that students' proportional reasoning looks the same[16] after they learned the formal concept of proportion via situations in grade 6 and after they re-learned the concept as the function y=ax in grade 7. But the results of problem 2 show the change in their reasoning from grade 6 to grade 7. Q3 in problem 2, (see Graph 4) shows that grade 6 students' proportion of correct answers was higher[17] than in grade 7, but is the same as grade 9. Graph 5 indicates that, to get a correct answer, grade 6 students' solving methods of problem 1 and of Q3 were more different than grade 7 students'. Q2 in problem 2, (see Graph 3) shows that many grade 7 students still recognized this situation as dealing with proportions. Graph 6 show that half of them could not write a correct answer to Q3. The difference between problem 1 and problem 2 is that problem 2 was posed via a real situation. This result suggests that many grade 7 students, in the process of reconstructing the concept of proportion as a function, become lost when applying the concept of proportion to the real world. Indeed, Graphs 1 and 2 for Q1 show that after learning proportion, grade 6 students could describe and analyze the situation itself exactly, while grade 7 students, having re-learned proportion as a function, could not.

[13]In the current curriculum, this redefinition of function is taught in Grade 10. Examples were collected in the former curriculum.

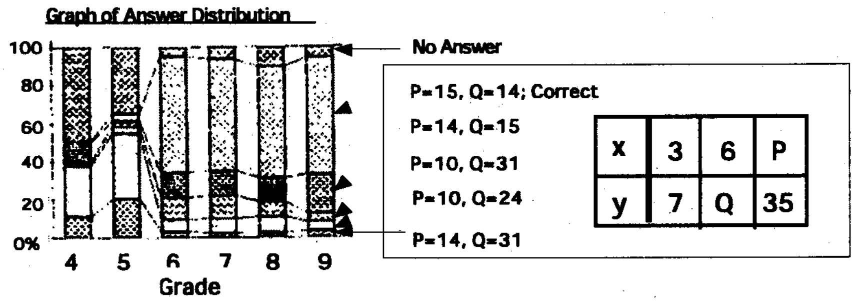
[14]This data was collected in a down town area of big cities and each grade's population was larger than 150 people. They had already learned each grade content of function or functional thinking area in the national curriculum.

[15]This problem is the same as a problem in the Second International Mathematics Study.

[16] The probability of no difference is 0.6. There is no significant difference.

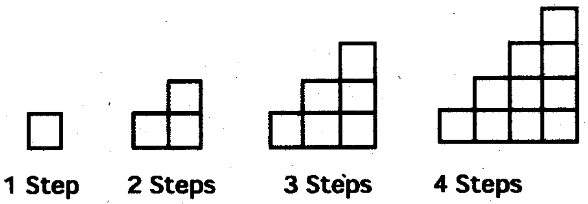
[17]The probability of no difference is 0.00015. There is a significant difference.

Problem 1 In the right table, if y is in proportion to x, then select the pair which is appropriate for P and Q in the table.

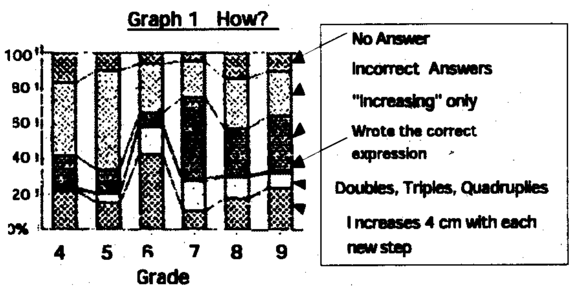


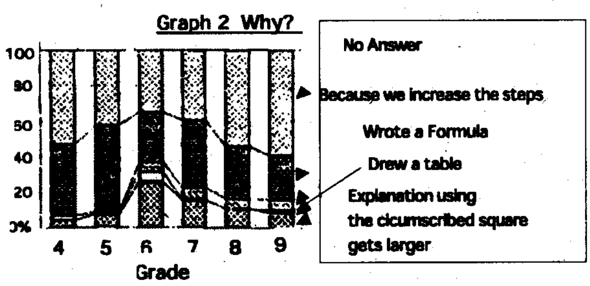
x	3	6	P
y	7	Q	35

Problem 2. Let's make stairs using squares with sides 1 cm as follows.

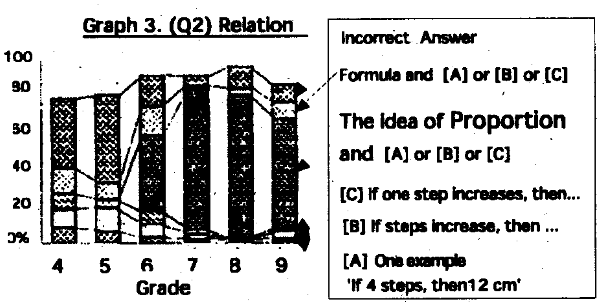


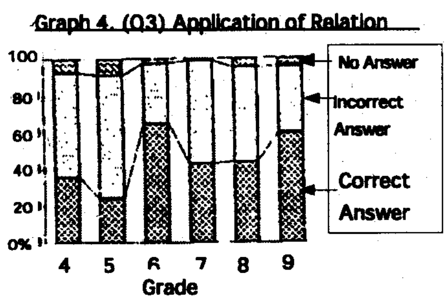
Q1. How does the perimeter change as the number of steps increases? Why do you think so?



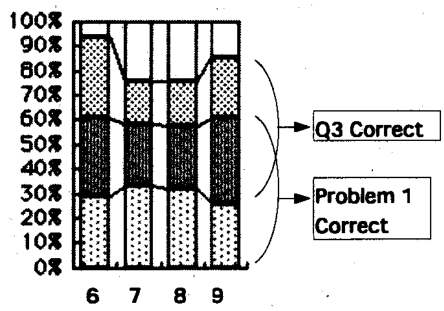


Q2. How can we relate the number of steps and the perimeter?
Q3. What is the perimeter if there are ten steps?

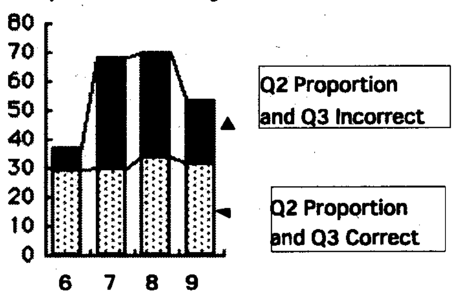




Graph 5. Cross-Analysis of Problem 1 and Q3



Graph 6. Cross-Analysis of Q2 and Q3



One interpretation of these results is that many students who already know the concepts of proportion and have experience dealing with y=ax in the context of real situations, can not assimilate the function y=ax in the context of algebraic discussions. But Q3 of problem 2 shows that, in grade 9, many students are again able to find the answer. Thus, it can be interpreted that students in grade 9 had accommodated their knowledge.

Example 2. Students' thinking is still viable until they meet a non-viable situation; We read tables as representing covariation and correspondence. In the Japanese curriculum, functions are taught using correspondence in grade 7 to assist students to level 3. Teachers begin to call a table of function a 'Correspondence Table' when teaching correspondence. But the results of problem 3 show that students do not change their thinking until grade 9 during which they learn the function $y=ax^2$, which is not easy to read covariationally. Indeed, in spite of students having been taught the y=ax table as correspondence from grade 7, many students continued to read the table covariationally until they were taught $y=ax^2$.

Problem 3. Write what you can find from the following tables.

(1)

x	1	2	3	4
y	4	8	12	16

(2)

x	1	2	3	4
y	2	8	18	32

Result of (1)

Grade	6	7	8	9
Covariation	42%	48%	49%	35%
Correspondence	24%	14%	16%	35%
Both	8%	10%	11%	11%

Result of (2)

Grade	6	7	8	9
Covariation	27%	15%	11%	10%
Correspondence	37%	21%	20%	50%
Both	0	0	2%	3%

Discussion

Examples 1 and 2 show that teaching supports students' transitions to level 3. It would be better to interpret the development of students' thinking from a lower level to a higher level as resembling an expanding equilibration rather than a monotonous increase. Furthermore, the above examples reflect the features of van

Heile's Levels. Indeed, based on these features, we can more critically interpret these examples.

Critical Explanation of Example 1. In order to explain example 1, hierarchy and duality of the levels of function in the context of Japanese curriculum must be discussed. To move students to level 2, teachers teach rules, for example proportion, using arithmetic on tables via real situations which were represented on level 1 using everyday language. To move students to level 3, teachers teach functions using algebra and geometry via rules which were represented on level 2 using arithmetic language with tables. Arithmetic language claims to move students to level 3, but in the case of everyday language, although students use it, they do not need to use everyday language in order to learn about functions algebraically and geometrically.

The notions of hierarchy and duality support a clearer explanation of example 1. Indeed, in grade 6, to move to level 2, teachers teach the concept of proportion using tables via real situations on level 1. And in grade 7, to move to level 3, teachers teach functions of the form y=ax using equations and graphs via the concept of proportion which was represented in arithmetic tables. Therefore, in problem 1, which was only represented with a table, there is no difference between the results in grade 6, 7, 8 and 9. But problem 2 was represented with a situation. Because grade 7 students had not learned the *function* y=ax with situations using everyday language, they overlooked/lost proportional reasoning in the situation.

Critical Explanation of Example 2. If we suppose that student thinking can be changed depending on the context of the teaching situation, example 2 can be more fully explained. Despite the fact that teachers explain correspondence using tables of y=ax, the students did not change their reasoning. But when teachers explained correspondence on tables of $y=ax^2$, the students did change their reasoning. Indeed, in the case of the table for y=ax, students learned covariance in grade 6 (level 2), and as we already saw in example 1, it was not changed in grade 7. If they know covariance, e.g. the that first difference of y=ax and y=ax+b is constant, then they can make a table. So, this knowledge is still viable in grades 7 and 8, during which they move to level 3. But in the case of $y=ax^2$ taught in grade 9, the first difference is not constant. In order to make a table, since students could not use the first difference they had to use correspondence. Thus, the quadratic function $y=ax^2$ provided a context that helped students understand the notion of correspondence.

Table 1, Examples 1 and 2 indicate that the levels of function include all four features of van Heile's Levels. Furthermore, it has been implyed that students' thinking is better characterized as an expanding equilibration

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